Objectives:

- Find limits of rational functions in cases where we can't substitute
- Find limits of piecewise functions
- Define and use the Squeeze Theorem

We saw last time that if $f(x)$ is a rational function and a is in the domain of f, then $\lim_{x \to a} f(x) = f(a)$. If a is a number not in the domain of $f(x)$, trying to substitute leads to dividing by zero.

1. If trying to plug a into $f(x)$ leads to "$\frac{\text{non-zero}}{0}$", then there is a vertical asymptote at a. This means the one-sided limits can be ∞ or $-\infty$.

   Example $\lim_{x \to 2} \frac{x+5}{x-2}$
   If we try to substitute, we get "$\frac{7}{0}$".
   Lefthand: $x < 2$, so $\lim_{x \to 2^-} \frac{x+5}{x-2} =$ "$\frac{7}{\text{tiny negative}}$" $= \frac{+}{-}\infty = -\infty$.
   Righthand: $x > 2$ so, $\lim_{x \to 2^+} \frac{x+5}{x-2} =$ "$\frac{7}{\text{tiny positive}}$" $= \frac{+}{+}\infty = \infty$.
   Since $\infty \neq -\infty$, the limit of $\frac{x+5}{x-2}$ as x goes to 2 does not exist.

   Example $\lim_{x \to 0} \frac{x+1}{x^2}$
   Attempting to substitute gives "$\frac{1}{0}$".
   Lefthand: "$\frac{1}{\text{tiny positive}}$" $= +\infty$.
   Righthand: "$\frac{1}{\text{tiny positive}}$" $= +\infty$.
   So, $\lim_{x \to 0} \frac{x+1}{x^2} = \infty$.

2. If trying to plug a into $f(x)$ leads to "$\frac{0}{0}$", the limit is indeterminate. There are a few strategies we can try:

   (a) Factor and Cancel

   Example $\lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{(x+2)(x-2)}{x-2} = \lim_{x \to 2} x+2 = 4$

   (b) Combine Fractions

   Example $\lim_{x \to 2} \frac{\frac{1}{x} - \frac{1}{2}}{x-2} = \lim_{x \to 2} \frac{\frac{2}{2x} - \frac{x}{2x}}{x-2} = \lim_{x \to 2} \frac{\frac{2-x}{2x}}{x-2} = \lim_{x \to 2} \frac{-(x-2)}{2x}\left(\frac{1}{x-2}\right) = \lim_{x \to 2} \frac{1}{2x} = \frac{1}{4}$

(c) Multiply by the Conjugate

Example $\lim_{x\to 4}\frac{\sqrt{x}-2}{x-4} = \lim_{x\to 4}\frac{\sqrt{x}-2}{x-4}\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right) = \lim_{x\to 4}\frac{x-4}{(x-4)(\sqrt{x}+2)} = \lim_{x\to 4}\frac{1}{\sqrt{x}+2}$

$$= \frac{2}{\sqrt{4}+2}$$

Example Let's calculate a limit that can't be approximated numerically on your calculator!

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\left(\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\right) = \lim_{t\to 0}\frac{(t^2+9)-9}{(t^2)(\sqrt{t^2+9}+3)} = \lim_{t\to 0}\frac{t^2}{(t^2)\sqrt{t^2+9}}$$

$$= \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{6}$$

Piecewise Functions

Not all functions are this nice! Piecewise functions require careful thinking about limit definitions:

$$f(x) = \begin{cases} -\sqrt{9+x} & -9 < x < -5 \\ 100 & x = -5 \\ x+3 & -5 < x \le 0 \\ x^2 & 0 < x \end{cases}$$

Example 1. $\lim_{x\to -5} f(x)$: $f(-5) = 100$, but this tells us nothing about the limit.

Lefthand side: Values "just less than" -5. These are located in $x < -5$, so
$\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} -\sqrt{9+x} = -\sqrt{9-5} = -\sqrt{4} = -2$

Righthand side: Values "just more than" -5, so
$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} x+3 = -5+3 = -2$

Since the one-sided limits agree, $\lim_{x\to -5} f(x) = -2$.

Example 2. $\lim_{x\to 0} f(x)$:

Lefthand side: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x+3 = 3$

Righthand side: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^2 = 0$

One-sided limits do not agree, so $\lim_{x\to 0} f(x)$ D.N.E.

Example 3. $\lim_{x\to -3} f(x)$: All values "near" -3 are in $-5 < x \le 0$, so $\lim_{x\to -3} f(x) = \lim_{x\to -3} x+3 = 0$

Don't forget $|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$.

Absolute value functions require just as much caution as any other piecewise function.

Example $\lim_{x\to 0} g(x)$ where $g(x) = \dfrac{x}{|x|}$:

To understand what's going on, we want to write this as $g(x) = \begin{cases} x/(-x) & x < 0 \\ x/x & x > 0 \end{cases} = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

It's worth noting that 0 is not in the domain of $g(x)$.

LHS: $\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} -1 = -1$ RHS: $\lim_{x\to 0^+} g(x) = \lim_{x\to\to 0^+} 1 = 1$

So $\lim_{x\to 0} g(x)$ D.N.E.

The Squeeze Theorem:

If $f(x) \leq g(x)$ for all x near a, then, even if $f(a) > g(a)$, we would expect that:

$$\lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$$

From this reasonable fact, we can deduce:

The Squeeze Theorem (a.k.a. Sandwich Theorem):

If $h(x) \leq f(x) \leq g(x)$ for all x near a (not necessarily for $x = a$),

then $\lim_{x\to a} h(x) \leq \lim_{x\to a} f(x) \leq \lim_{x\to a} g(x)$

The Squeeze Theorem is useful for finding limits of weird functions by "squeezing" them with more cooperative functions:

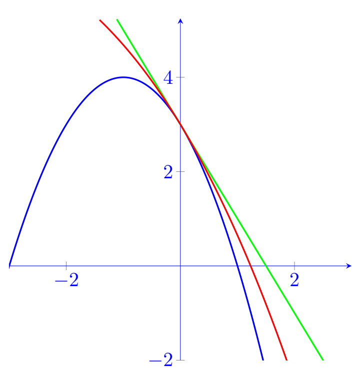
Example Let $f(x)$ be a mystery function. The only thing we know about f is $3 - 2x - x^2 \leq f(x) \leq -2x + 3$ for all $x \neq a$. Find $\lim_{x\to 0} f(x)$.

We know that $3 - 2x - x^2 \leq f(x) \leq -2x + 3$ for all $x \neq a$ so we can use the squeeze theorem with $h(x) = 3 - 2x - x^2$ and $g(x) = -2x + 3$. The squeeze theorem tells us that

$$\lim_{x\to 0} 3 - 2x - x^2 \leq \lim_{x\to 0} f(x) \leq \lim_{x\to 0} -2x + 3.$$

We can compute the limits of the polynomials so we have

$$3 \leq \lim_{x\to 0} f(x) \leq 3, \text{ so } \lim_{x\to 0} f(x) = 3$$



Example Find $\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right)$.

We might be tempted to split up the product here, but $\lim_{t\to 0} \sin\left(\frac{1}{t}\right)$ D.N.E.

Instead, let's use the fact that $-1 \le \sin\left(\frac{1}{t}\right) \le 1$ to squeeze our function.

$$(-1)t^2 \le t^2 \sin\left(\frac{1}{t}\right) \le (1)t^2$$

$$\lim_{t\to 0}(-1)t^2 \le \lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) \le \lim_{t\to 0}(1)t^2$$

$$0 \le \lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) \le 0$$

So, $\lim_{t\to 0} t^2 \sin\left(\frac{1}{t}\right) = 0$